On the Compressibility of Highly Repetitive Sequences

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Introduction

The world is drowning in data! In recent years, we have been deluged by a torrent of data from a variety of increasingly data-intensive applications, including databases, scientific computations, graphics, entertainment, multimedia, sensors, web applications, and email. NASA's Earth Observing System project, the core part of the Earth Science Enterprise (formerly Mission to Planet Earth), produces petabytes (10^{15} bytes) of raster data per year [148]. A petabyte corresponds roughly to the amount of information in one billion graphically formatted books. The online databases of satellite images used by Microsoft TerraServer (part of MSN Virtual Earth) [325] and Google Earth [180] are multiple terabytes (10^{12} bytes) in size. Wal-Mart's sales data warehouse contains over a half petabyte (500 terabytes) of data. A major challenge is to develop mechanisms for processing the data, or else much of the data will be useless.

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- ... are we drowning in information?

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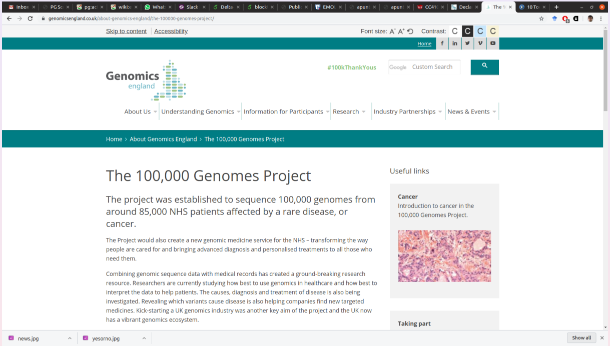
- Myriad genomes of the same species.



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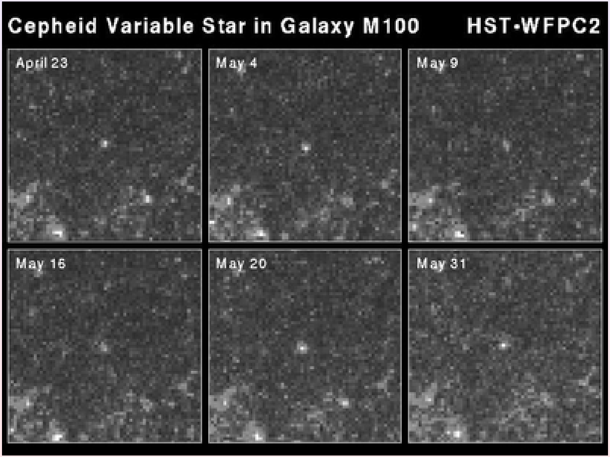

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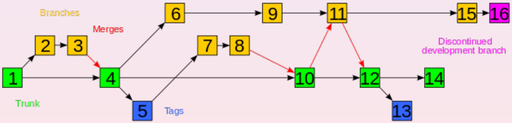

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  - Using Lempel-Ziv compression.

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We will focus on sequence data, and on the following questions:

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- Can we access the text efficiently within that space?

Shannon's entropy?

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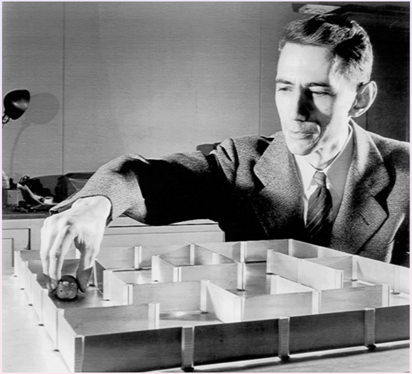

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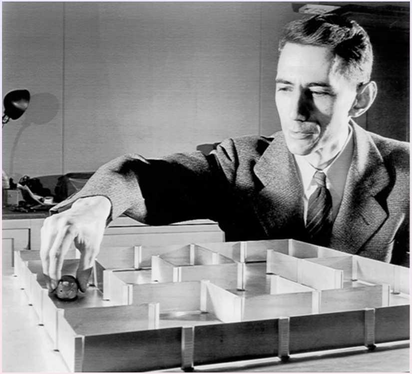

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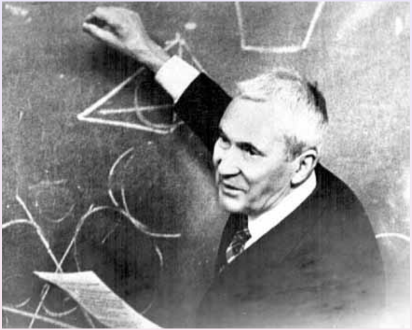

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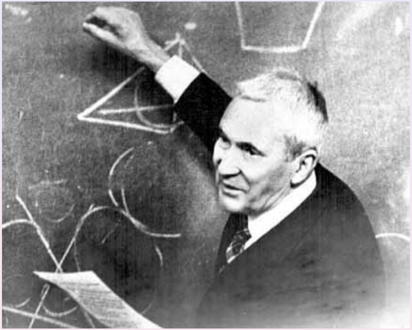

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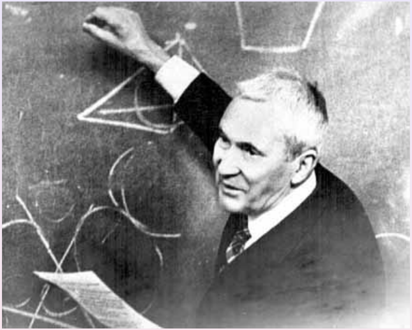

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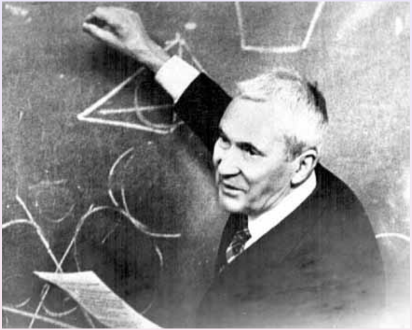

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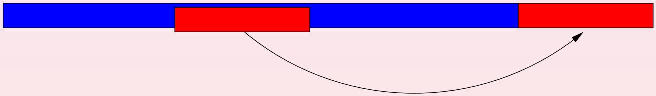

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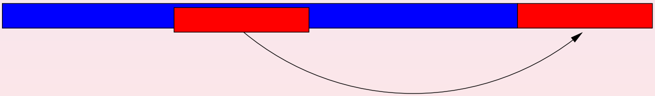

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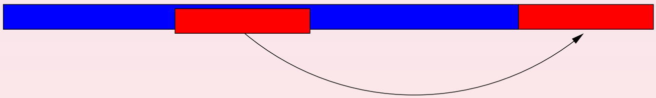

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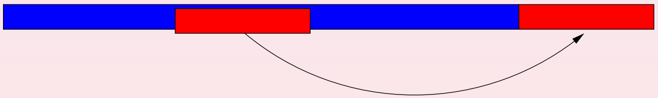

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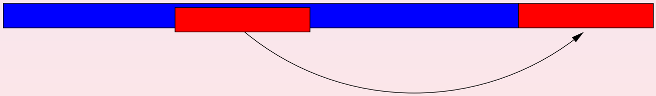

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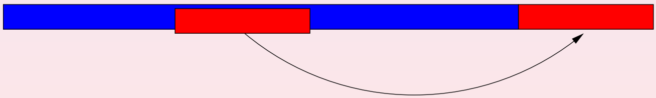
- The number z of phrases is the Lempel-Ziv complexity.



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Output: $(0, 0, a)$

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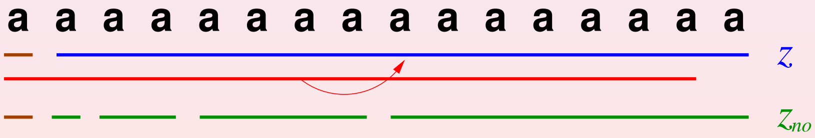

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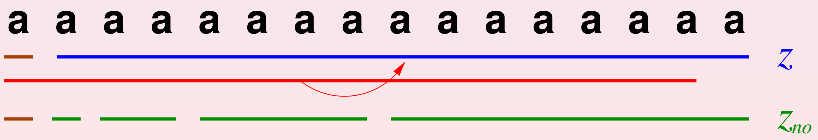

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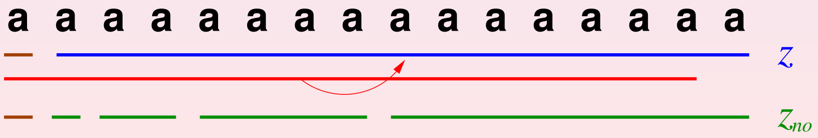

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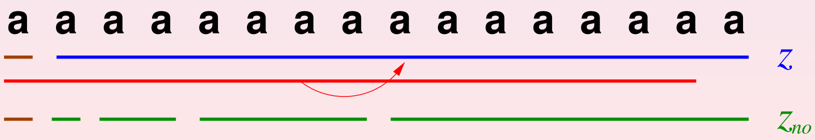

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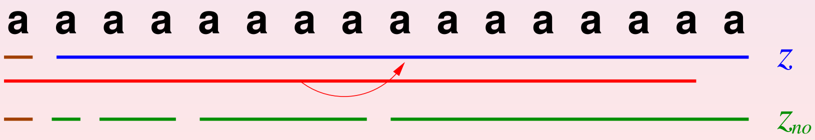

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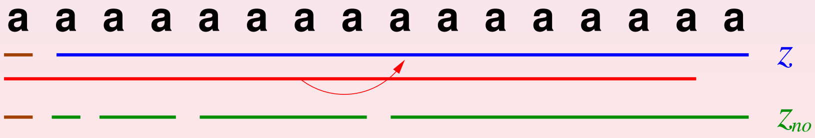

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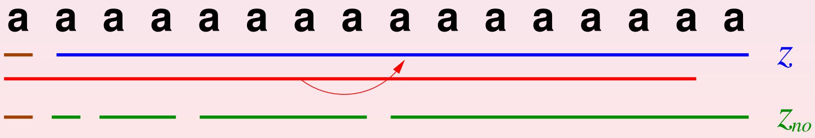
- The base of practical compressors like LZ77 and LZ78, with immense success.



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- It may double upon a single character edit on T [Akagi, Funakoshi, Inenaga 2022].

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- z_e is reasonably close to z in practice.

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a l a b a r a l a l a b a r d a $

$z = 7$

$z_{no} = 7$

a l a b a r a l a l a b a r d a $

$z_e = 7$

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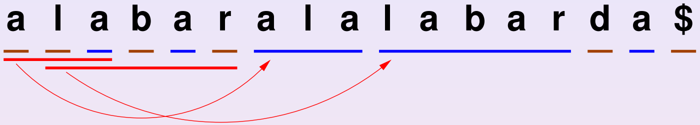

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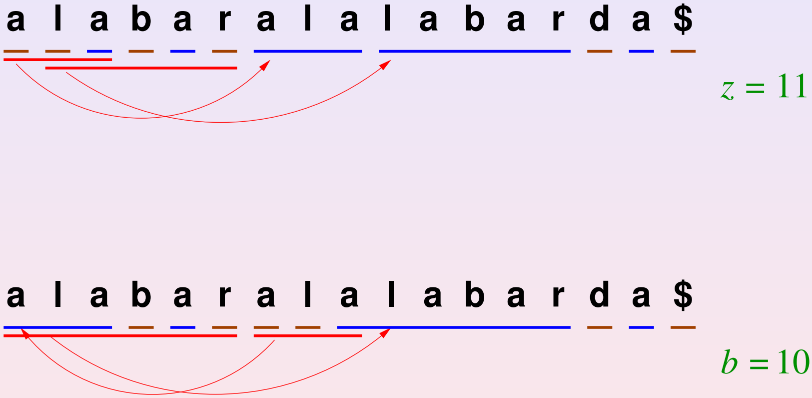
Bidirectional macro schemes



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- No interesting approximations except for z.

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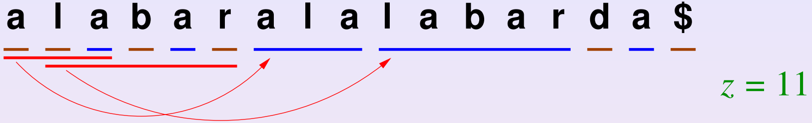

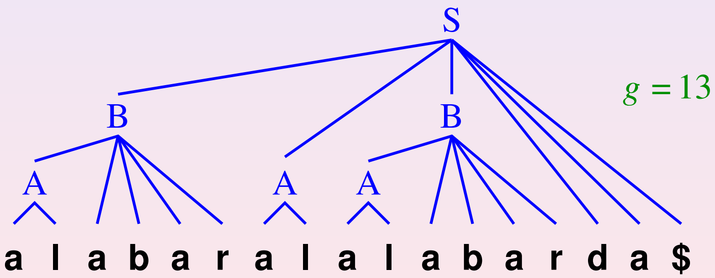

A → a l   B → A a b a r   S → B A B d a $

The relation between grammars and Lempel-Ziv

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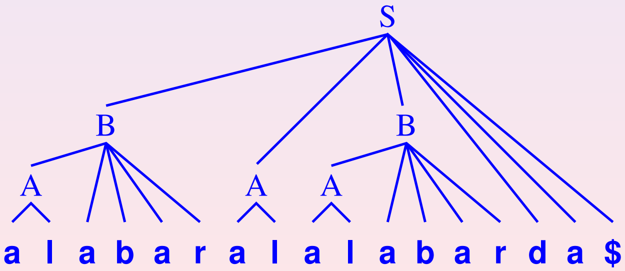

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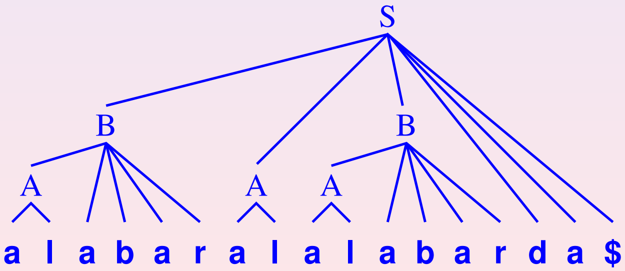

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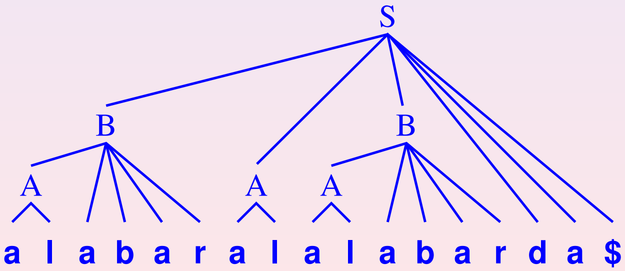

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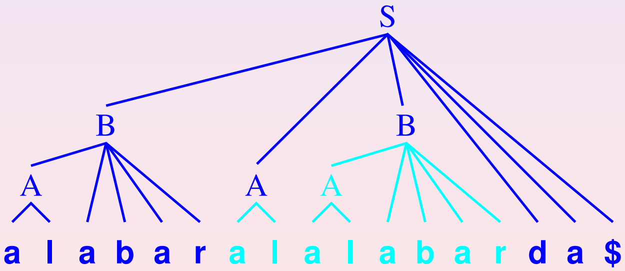

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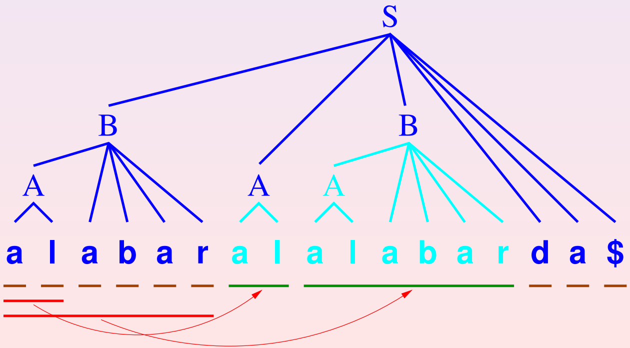

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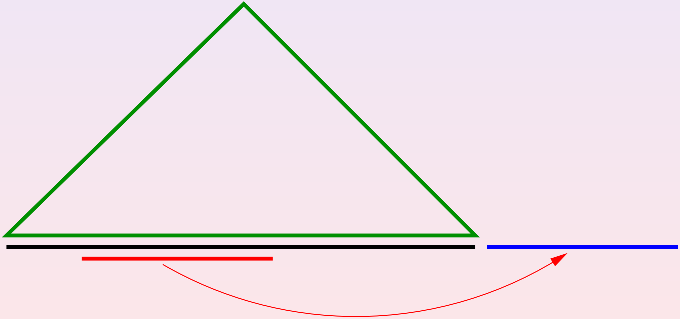

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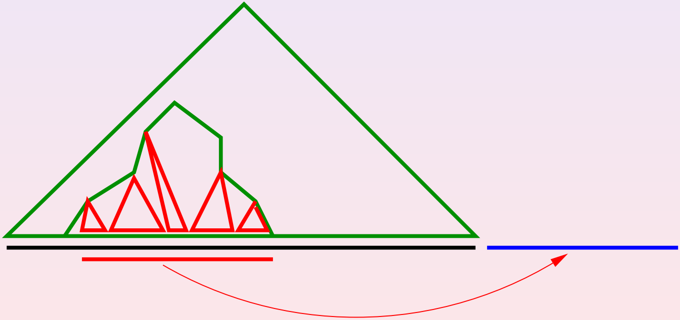

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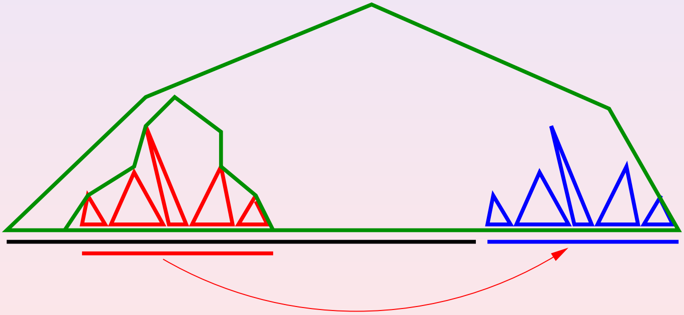

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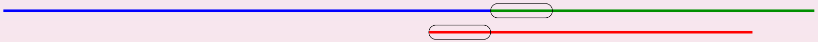

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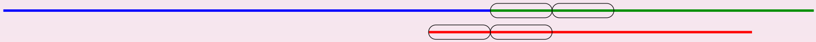

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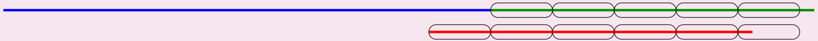

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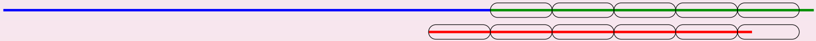

Accessing T in compressed space

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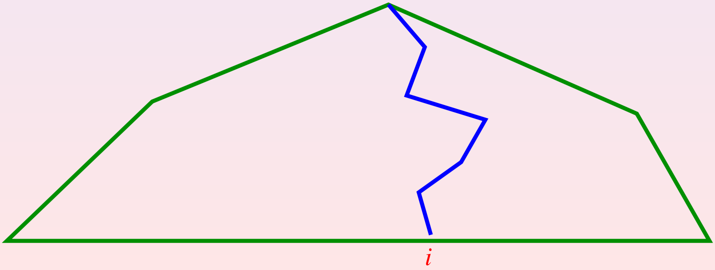

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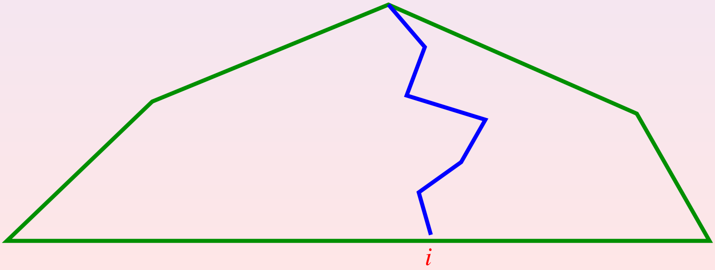

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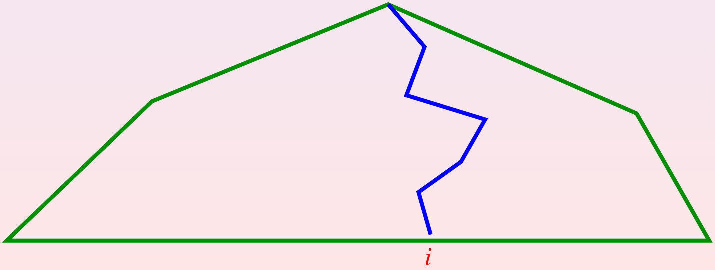

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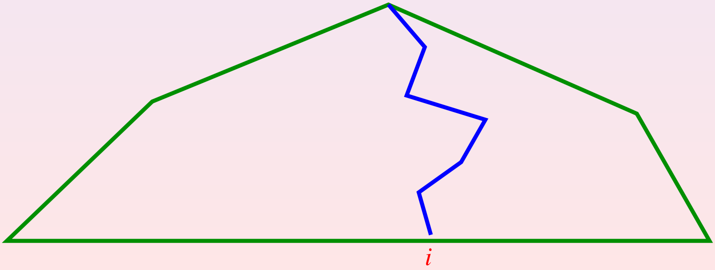

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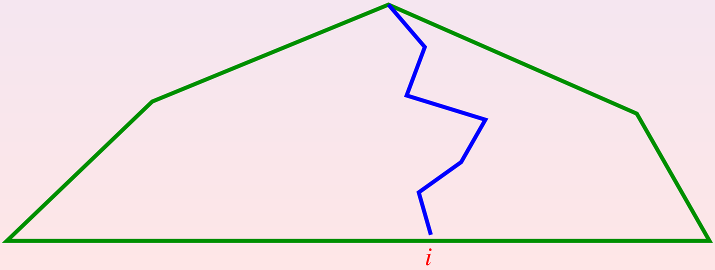

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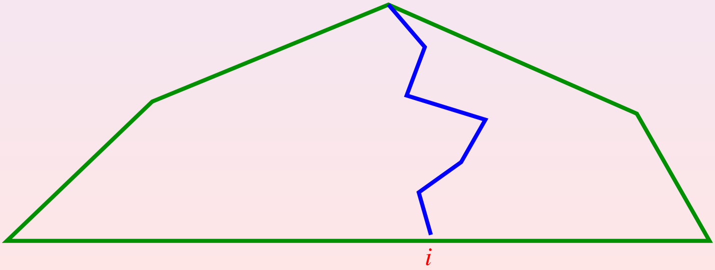

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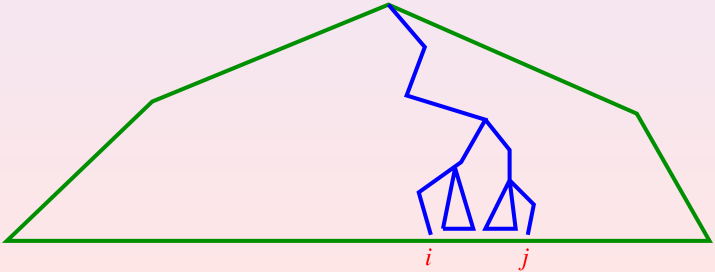

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- Because every grammar can be made binary and balanced within the same asymptotic size [Ganardi, Jez, Lohrey 2020].

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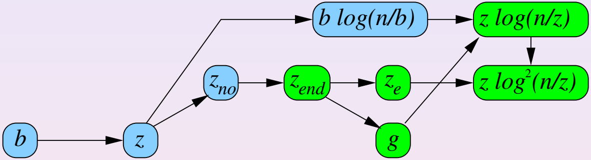
- It is parsed into $z_{no} = O(q + \log k_1) = O(\log k_1)$ phrases.
- Its grammar requires size $\Omega(\log^2 k_1 / \log\log k_1)$.

Recap



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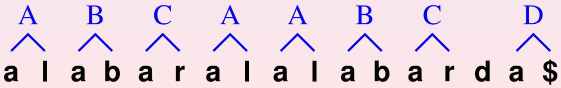

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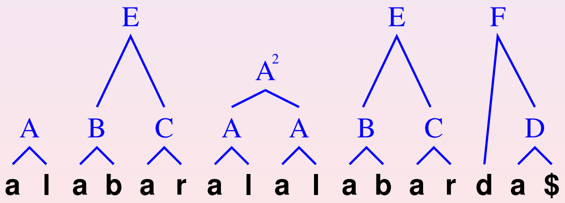

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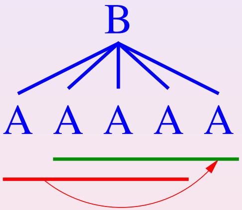
- They (easily) prove $z = O(g_{rl})$, thus the bound we already saw, $z = O(b \log(n/b))$.



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- Christiansen et al. [2020] generalized the result to run-length grammars.
- So one can access any $T[i..i+\ell]$ within $O(g_{rl})$ space in time $O(\log n+\ell)$.

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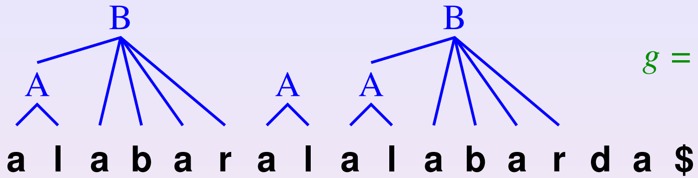

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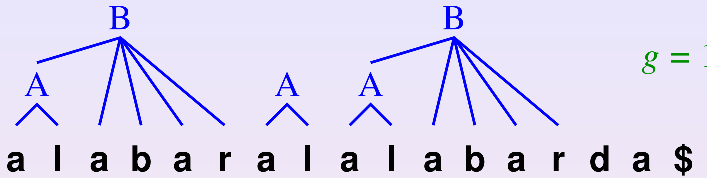

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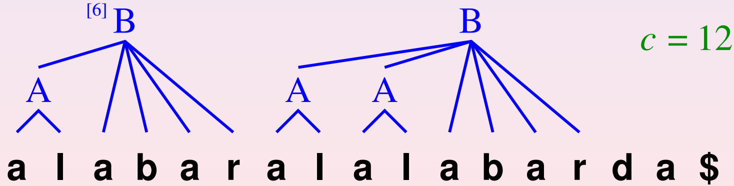

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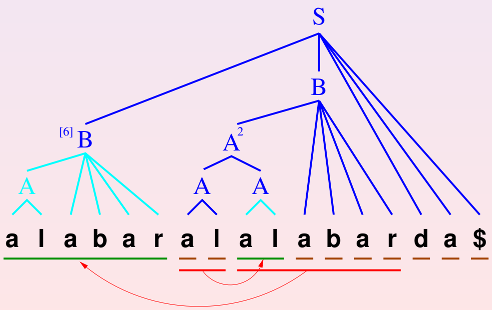

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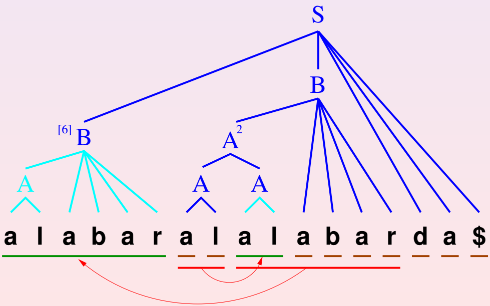

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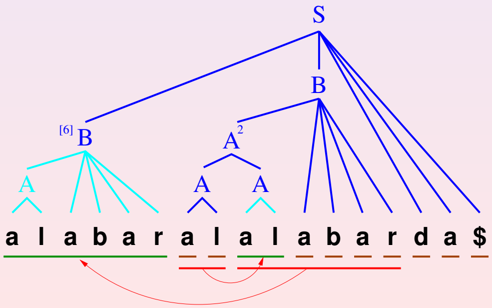

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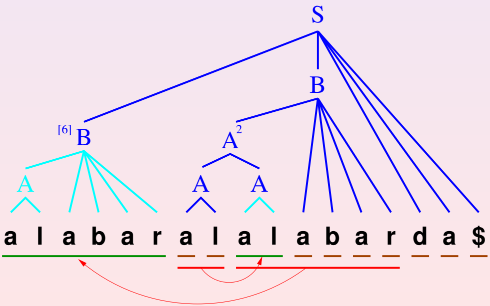

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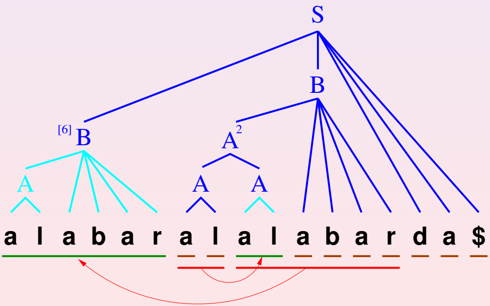

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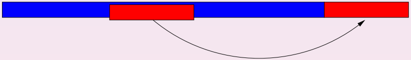

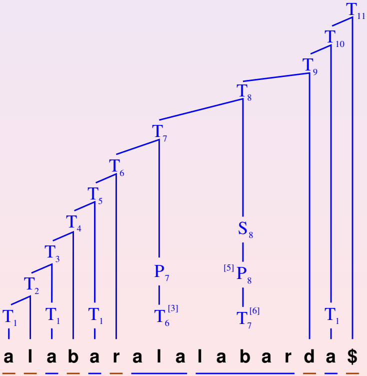

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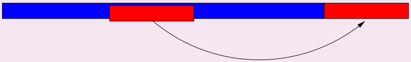

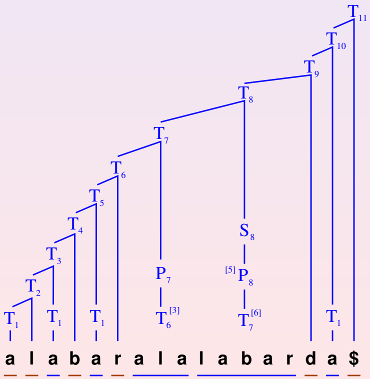

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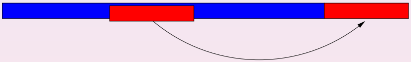

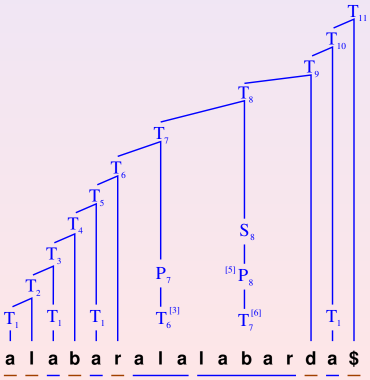

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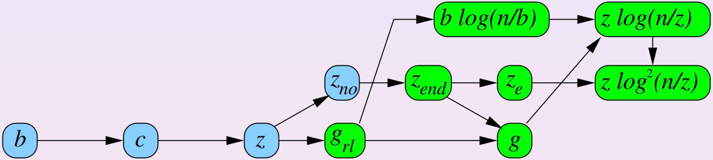

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  - For repetitive texts, r is small [Mäkinen et al. 2008].

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```
$ a l a b a r a l a l a b a r d a
a $ a l a b a r a l a l a b a r d
a b a r a l a l a b a r d a $ a l
a b a r d a $ a l a b a r a l a l
a l a b a r a l a l a b a r d a $
a l a b a r d a $ a l a b a r a l
a l a l a b a r d a $ a l a b a r
a r a l a l a b a r d a $ a l a b
a r d a $ a l a b a r a l a l a b
b a r a l a l a b a r d a $ a l a
b a r d a $ a l a b a r a l a l a
d a $ a l a b a r a l a l a b a r
l a b a r a l a l a b a r d a $ a
l a b a r d a $ a l a b a r a l a
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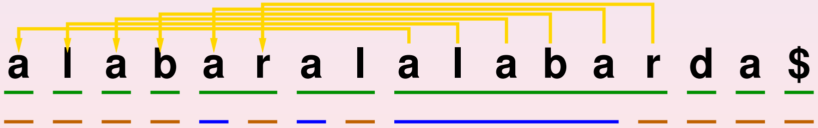

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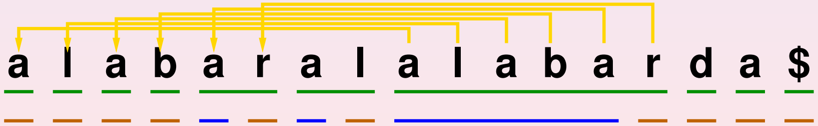

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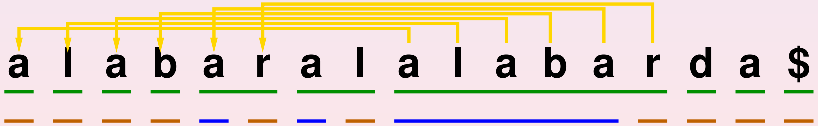

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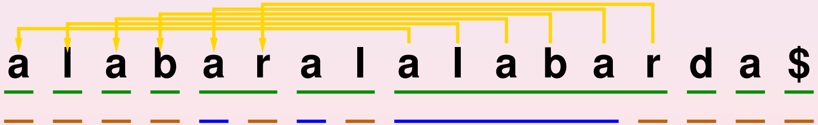

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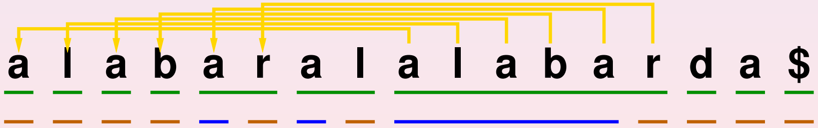

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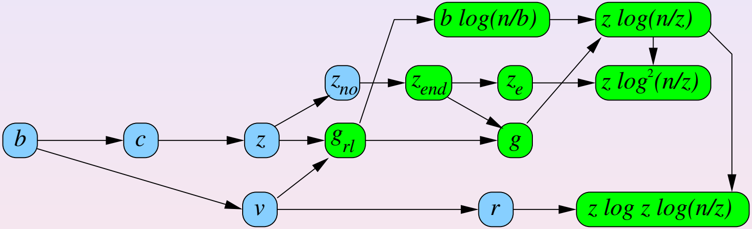
Recap



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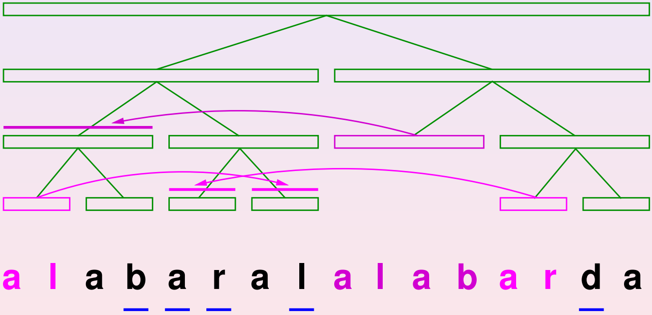
Accessing with attractors

Block-tree-like structure



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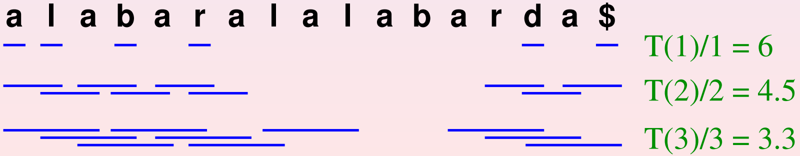

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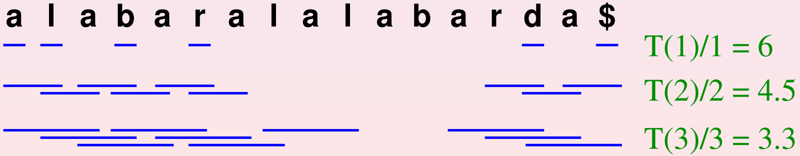

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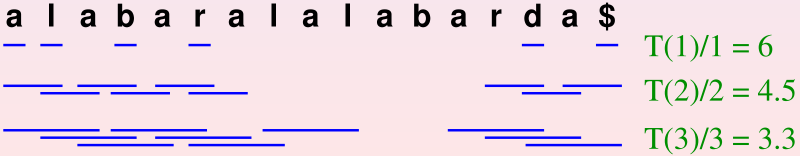

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Some upper bounds in terms of δ

- $z = O(\delta \log(n/\delta))$ [Raskhodnikova et al. 2013], so this is reachable.
- Block trees can be made of size $O(\delta \log(n/\delta))$, so direct access within that size [Kociumaka, N., Prezza 2020].
- Further, $g_{rl} = O(\delta \log(n/\delta))$ [Kociumaka, N., Prezza 2021].
  - Similar to the proof for $g = O(b \log(n/b))$.
  - "Pausing" long symbols to avoid them growing too fast.
  - This cannot be achieved with g.
- It also holds $r = O(\delta \log \delta \log(n/\delta))$ [Kociumaka & Kempa 2019] and $z_e = O(\delta \log^2(n/\delta))$ [Kempa & Saha 2022].

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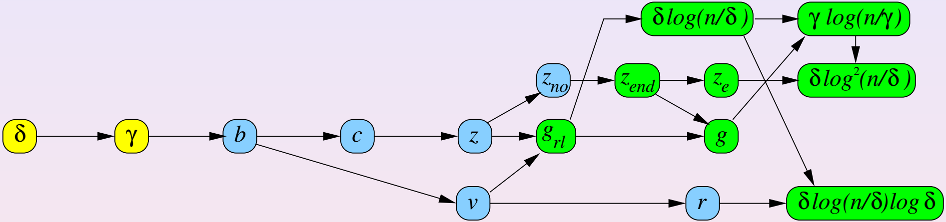
  - Using a generalized variant of the above family.

Recap



- $z \approx v \approx$ 1.5–2.5 $\cdot\, \delta$
- $g \approx$ 3–6 $\cdot\, \delta$
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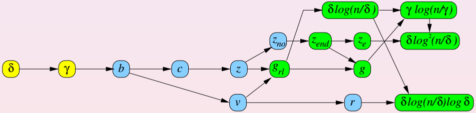

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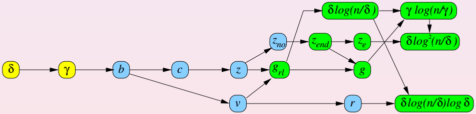

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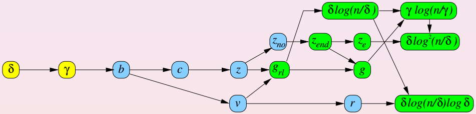

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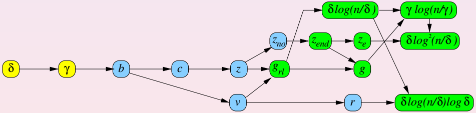

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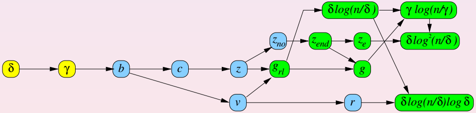

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  - Is at least δ a final lower bound?

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  - If all the rules have right-hand sides of the same length...
  - Then the string has $\gamma = O(\log n)$ [Shallit 2020].

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  - Maybe ℓ captures better some repetitive structured texts?

Thanks for your attention!